Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 18034–18047.
DOI: 10.3934/math.2024879
Received: 20 December 2023
Revised: 13 April 2024
Accepted: 26 April 2024
Published: 28 May 2024

Research article

Bounding coefficients for certain subclasses of bi-univalent functions related to Lucas-Balancing polynomials

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Abstract: In this paper, we introduced two novel subclasses of bi-univalent functions, $\mathcal{M}_\Sigma(\alpha, \mathcal{B}(x,\xi))$ and $\mathcal{H}_\Sigma(\alpha,\mu,\mathcal{B}(x,\xi))$, utilizing Lucas-Balancing polynomials. Within these function classes, we established bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, addressing the Fekete-Szegö functional problems specific to functions within these new subclasses. Moreover, we illustrated how our primary findings could lead to various new outcomes through parameter specialization.

Keywords: balancing polynomial; Lucas-Balancing polynomials; bi-univalent functions; analytic functions; Taylor-Maclaurin coefficients; Fekete-Szegö functional
Mathematics Subject Classification: 30C45

1. Introduction

Let $\mathcal{A}$ denote the set of all functions f, which are analytic in the open unit disk $\mathbb{U} = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$ and has a Taylor-Maclaurin series expansion given by

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \ (\xi \in \mathbb{U}). \tag{1.1}$$

Additionally, functions in $\mathcal{A}$ are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the set of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. For $f, g \in \mathcal{A}$, we say f is subordinate to g if there exists a Schwarz function $h(\xi)$ such that $h(0) = 0$, $|h(\xi)| < 1$, and $f(\xi) = g(h(\xi))$ for $\xi \in \mathbb{U}$. Symbolically, this relationship is denoted as $f \prec g$ or $f(\xi) \prec g(\xi)$ for $\xi \in \mathbb{U}$. Miller et al. [1] state that if the function g is univalent in $\mathbb{U}$, then the subordination can be equivalently expressed as $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. The Koebe one-quarter theorem [2] guarantees the existence of an inverse function, denoted as f^{-1}, for any function $f \in \mathcal{S}$, satisfying the following conditions:

$$f^{-1}(f(\xi)) = \xi, \quad (\xi \in \mathbb{U}),\ f\left(f^{-1}(w)\right) \ = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4}), \tag{1.2}$$

where,

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.3}$$

A function $f \in \mathcal{A}$ is considered bi-univalent within the domain $\mathbb{U}$ if both the function f and its inverse f^{-1} are one-to-one within $\mathbb{U}$. Let Σ denote the set of bi-univalent functions within the domain $\mathbb{U}$, as specified by Eq (1.1).

Here, we present several examples of functions belonging to the class Σ which have significantly reinvigorated the study of bi-univalent functions in recent years:

$$f_1(\xi) = \frac{\xi}{1-\xi} \quad f_2(\xi) = -\log(1-\xi) \quad \text{and} \quad f_3(\xi) = \frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right),$$

with their respective inverses

$$f_1^{-1}(w) = \frac{w}{1+w} \quad f_2^{-1}(w) = \frac{e^w - 1}{e^w} \quad \text{and} \quad f_3^{-1}(w) = \frac{e^{2w} - 1}{e^{2w} + 1}.$$

However, the Koebe function denoted by $K(\xi) = \frac{\xi}{(1-\xi)^2}$ does not belong to the class Σ because it maps the open unit disk $\mathbb{U} \subset \mathbb{C}$ to $\mathrm{K}(\mathbb{U}) = \mathbb{C} \backslash (-\infty, -\frac{1}{4}]$, which does not include $\mathbb{U}$.

The most significant and thoroughly investigated subclasses of $\mathcal{S}$ are the class $\mathcal{S}^*(\delta)$ of starlike functions of order $\delta \in [0, 1)$ and the class, $\mathcal{K}(\delta)$ of convex functions of order δ in the open unit disk $\mathbb{U}$, which are respectively defined by

$$\mathcal{S}^*(\delta) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{\frac{\xi f'(\xi)}{f(\xi)}\right\} > \delta, (\xi \in \mathbb{U};\ 0 \leq \delta < 1) \right\}$$

and

$$\mathcal{K}(\delta) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{1 + \frac{\xi f''(\xi)}{f'(\xi)}\right\} > \delta, (\xi \in \mathbb{U};\ 0 \leq \delta < 1) \right\}.$$

Fekete and Szegö [3] established a fundamental finding regarding the maximum value of $\left|a_3 - \eta a_2^2\right|$ within the class of normalized univalent functions defined in (1.1), where η is a real parameter. Subsequent studies have expanded upon this, investigating $\left|a_3 - \eta a_2^2\right|$ for various classes of functions defined in terms of subordination. Numerous authors have made significant strides in establishing tight coefficient bounds for diverse subclasses of bi-univalent functions, often intertwined with specific polynomial families (see [4–13]).

In [14], Behera and Panda introduced a novel integer sequence called Balancing numbers. These numbers are defined by the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$, with initial values $B_0 = 0$ and $B_1 = 1$. Several researchers have explored these new number sequences, leading to the establishment of various generalizations. Comprehensive information on Lucas-Balancing numbers and their extensions can be found in [15–23]. One notable extension is the Lucas Balancing polynomial, which is recursively defined as follows:

Definition 1.1 (Lucas-Balancing Polynomials, [24])**.** *For any complex number x and integer $n \geq 2$, Lucas-Balancing polynomials are defined recursively as follows:*

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \tag{1.4}$$

where the initial conditions are given by:

$$C_0(x) = 1, \quad C_1(x) = 3x. \tag{1.5}$$

Using the recurrence relation (1.4)*, we can derive the following expressions:*

$$C_2(x) = 18x^2 - 1 \quad C_3(x) = 108x^3 - 9x. \tag{1.6}$$

Lucas-Balancing polynomials, like other number polynomials, can be derived through certain generating functions. One such generating function is expressed as follows:

Lemma 1.1. *[24] The generating function for Balancing polynomials can be represented as*

$$\mathcal{B}(x,\xi) = \sum_{n=0}^{\infty} C_n(x)\xi^n = \frac{1-3x\xi}{1-6x\xi+\xi^2}, \tag{1.7}$$

where x is within the range $[-1,1]$, and ξ is in the open unit disk $\mathbb{U}$.

A recently published paper by Hussen and Illafe [25] employs a novel approach utilizing the linear combination of two distinct subclasses, starlike and convex functions, associated with Lucas-Balancing polynomials $\mathcal{N}_{\Sigma}^{\lambda}(\mathcal{B}(x,z))$. They aim to determine the Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$, while addressing the Fekete-Szegö functional inequality. In this paper, we extend this investigation by exploring alternative subclasses connected with Lucas-Balancing polynomials.

Lemma 1.2. *[2] Let Ω be the class of all analytic functions, and let $\omega \in \Omega$ with $\omega(\xi) = \sum_{n=1}^{\infty} \omega_n \xi^n$, $\xi \in \mathbb{D}$. Then,*

$$|\omega_1| \leq 1, \quad |\omega_n| \leq 1 - |\omega_1|^2 \quad for \quad n \in \mathbb{N}\backslash\{1\}.$$

2. Coefficient bounds of the class $\mathcal{M}_{\Sigma}(\alpha, \mathcal{B}(x,\xi))$

Embarking on our exploration, we aim to introduce and define a distinct class of bi-univalent functions. This novel subclass, denoted as $\mathcal{M}_{\Sigma}(\alpha, \mathcal{B}(x,\xi))$, will expand our understanding and contribute to the evolving landscape of mathematical analysis in the domain of bi-univalent functions.

Definition 2.1. *A function $f \in \Sigma$ given by* (1.1)*, with $\alpha \in [0,1]$ and $x \in (\frac{1}{2}, 1]$, is said to be in the class $\mathcal{M}_{\Sigma}(\alpha, \mathcal{B}(x,\xi))$ if the following subordinations are satisfied*

$$\frac{\xi f'(\xi)}{f(\xi)} + \alpha \frac{\xi^2 f''(\xi)}{f(\xi)} \prec \mathcal{B}(x,\xi) \tag{2.1}$$

and

$$\frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}\prec\mathcal{B}(x,w), \tag{2.2}$$

where the function $g(w)=f^{-1}(w)$ *is defined by* (1.3) *and* $\mathcal{B}(x,\xi)$ *is the generating function of the Lucas-Balancing polynomials given by* (1.7).

Example 2.1. *A bi-univalent function* $f\in\Sigma$ *is said to be in the class* $\mathcal{M}_\Sigma(0,\mathcal{B}(x,\xi))$, *if the following subordination conditions hold:*

$$\frac{\xi f'(\xi)}{f(\xi)}\prec\mathcal{B}(x,\xi) \tag{2.3}$$

and

$$\frac{wg'(w)}{g(w)}\prec\mathcal{B}(x,w), \tag{2.4}$$

where the function $g=f^{-1}$ *is defined by* (1.3).

Theorem 2.1. *Let* f *given by (1.1) be in the class* $\mathcal{M}_\Sigma(\alpha,\mathcal{B}(x,\xi))$. *Then,*

$$|a_2|\le\frac{|C_1(x)|\ \sqrt{|C_1(x)|}}{\sqrt{\left|(1+4\alpha)(C_1(x))^2-(1+2\alpha)^2C_2(x)\right|}}$$

and

$$|a_3|\le\frac{27x^3}{\left|9x^2(1+4\alpha)-(18x^2-1)(1+2\alpha)^2\right|}+\frac{3x}{2(1+3\alpha)}.$$

Proof. Given that $f\in\mathcal{M}_\Sigma(\alpha,\mathcal{B}(x,\xi))$, where $0\le\alpha\le1$, it follows from Eqs (2.1) and (2.2) that

$$\frac{\xi f'(\xi)}{f(\xi)}+\alpha\frac{\xi^2f''(\xi)}{f(\xi)}=\mathcal{B}(x,u(\xi)) \tag{2.5}$$

and

$$\frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}=\mathcal{B}(x,v(w)), \tag{2.6}$$

where $g(w)=f^{-1}(w)$ and $u,\ v\in\Omega$ are given to be of the form

$$u(\xi)=\sum_{n=1}^{\infty}c_n\,\xi^n\qquad\text{and}\qquad v(w)=\sum_{n=1}^{\infty}d_n\,w^n. \tag{2.7}$$

Utilizing Lemma 1.2 yields the following inequality

$$|c_n|\le1\ \text{ and }\ |d_n|\le1,\ \ n\in\mathbb{N}. \tag{2.8}$$

By replacing the expression of $\mathcal{B}(x,\xi)$ as defined in (1.7) into the respective right-hand sides of Eqs (2.5) and (2.6), we obtain

$$\mathcal{B}(x,u(\xi)) = 1 + C_1(x)c_1\xi + \left[C_1(x)c_2 + C_2(x)c_1^2\right]\xi^2 + \left[C_1(x)c_3 + 2C_2(x)c_1c_2 + C_3(x)c_1^3\right]\xi^3 + \cdots \quad (2.9)$$

and

$$\mathcal{B}(x,v(w)) = 1 + C_1(x)d_1w + \left[C_1(x)d_2 + C_2(x)d_1^2\right]w^2 + \left[C_1(x)d_3 + 2C_2(x)d_1d_2 + C_3(x)d_1^3\right]w^3 + \cdots . \quad (2.10)$$

Therefore, Eqs (2.5) and (2.6) become

$$\begin{aligned} &1 + a_2\xi + (2a_3 - a_2^2)\xi^2 + (a_2^3 - 3a_2a_3 + 3a_4)\xi^3 + \cdots \\ &+ \alpha\left[2a_2\xi + (6a_3 - 2a_2^2)\xi^2 + 2(a_2^3 - 4a_2a_3 + 6a_4)\xi^3\right] + \cdots \\ &= 1 + C_1(x)c_1\xi + \left[C_1(x)c_2 + C_2(x)c_1^2\right]\xi^2 + \left[C_1(x)c_3 + 2C_2(x)c_1c_2 + C_3(x)c_1^3\right]\xi^3 + \cdots \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} &1 - a_2w + (3a_2^2 - 2a_3)w^2 + (-10a_2^3 + 12a_2a_3 - 3a_4)w^3 + \cdots \\ &+ \alpha\left[-2a_2w + (10a_2^2 - 6a_3)w^2 + (-46a_2^3 + 52a_2a_3 - 12a_4)w^3\right] + \cdots \\ &= 1 + C_1(x)d_1w + \left[C_1(x)d_2 + C_2(x)d_1^2\right]w^2 + \left[C_1(x)d_3 + 2C_2(x)d_1d_2 + C_3(x)d_1^3\right]w^3 + \cdots . \end{aligned} \quad (2.12)$$

By equating the coefficients in Eqs (2.11) and (2.12), we obtain

$$(1+2\alpha)a_2 = C_1(x)c_1, \quad (2.13)$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = C_1(x)c_2 + C_2(x)c_1^2, \quad (2.14)$$

$$-(1+2\alpha)a_2 = C_1(x)d_1 \quad (2.15)$$

and

$$(3+10\alpha)a_2^2 - 2(1+3\alpha)a_3 = C_1(x)d_2 + C_2(x)d_1^2. \quad (2.16)$$

Utilizing Eqs (2.13) and (2.15) we derive the subsequent equations

$$c_1 = -d_1 \quad (2.17)$$

and

$$c_1^2 + d_1^2 = \frac{2(1+2\alpha)^2a_2^2}{\left(C_1(x)\right)^2}. \quad (2.18)$$

Moreover, utilizing Eqs (2.14), (2.16) and (2.18) results in

$$a_2^2 = \frac{(C_1(x))^3(c_2+d_2)}{2\left[(1+4\alpha)(C_1(x))^2-(1+2\alpha)^2C_2(x)\right]}. \tag{2.19}$$

Utilizing Lemma 1.2 and examining Eqs (2.13) and (2.17), we can deduce

$$|a_2|^2 \leq \frac{|C_1(x)|^3}{\left|(1+4\alpha)(C_1(x))^2-(1+2\alpha)^2C_2(x)\right|}, \tag{2.20}$$

consequently,

$$|a_2| \leq \frac{|C_1(x)|\sqrt{|C_1(x)|}}{\sqrt{\left|(1+4\alpha)(C_1(x))^2-(1+2\alpha)^2C_2(x)\right|}}. \tag{2.21}$$

Replacing the expressions for $C_1(x)$ and $C_2(x)$, as given in (1.5) and (1.6), respectively, into Eq (2.21) results in the following

$$|a_2| \leq \frac{3x\sqrt{3x}}{\sqrt{\left|9x^2(1+4\alpha)-(18x^2-1)(1+2\alpha)^2\right|}}.$$

By subtracting Eq (2.16) from Eq (2.14), we obtain

$$a_3 = a_2^2 + \frac{C_1(x)(c_2-d_2)}{4(1+3\alpha)}. \tag{2.22}$$

This results in the following inequality

$$|a_3| \leq |a_2|^2 + \frac{|C_1(x)|\,|c_2-d_2|}{4(1+3\alpha)}. \tag{2.23}$$

Applying Lemma 1.2, utilizing (1.5) and (1.6) we obtain

$$|a_3| \leq \frac{27x^3}{\left|9x^2(1+4\alpha)-(18x^2-1)(1+2\alpha)^2\right|} + \frac{3x}{2(1+3\alpha)}. \tag{2.24}$$

The proof of Theorem 2.1 is thus concluded.

□

3. Fekete-Szegö functional estimations of the class $\mathcal{M}_\Sigma(\alpha, \mathcal{B}(x,\xi))$

Within this section, the utilization of a_2^2 and a_3 serves as a crucial tool in establishing the Fekete-Szegö inequality applicable to functions belonging to $\mathcal{M}_\Sigma(\alpha, \mathcal{B}(x,\xi))$. This mathematical endeavor leverages these specific coefficients to derive insightful results within this functional space.

Theorem 3.1. *Let f given by (1.1) be in the class $\mathcal{M}_\Sigma(\alpha, \mathcal{B}(x,\xi))$. Then,*

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{3x}{2(1+4\alpha)} & \textit{if} \quad 0 \leq |h(\eta)| \leq \frac{1}{4(1+3\alpha)}, \\ 6x\,|h(\eta)| & \textit{if} \quad |h(\eta)| \geq \frac{1}{4(1+3\alpha)}, \end{cases}$$

where

$$h(\eta) = \frac{9x^2(1-\eta)}{2\left[9x^2(1+4\alpha) - (18x^2-1)(1+2\alpha)^2\right]}.$$

Proof. Based on Eqs (2.19) and (2.22), we obtain

$$\begin{aligned} a_3 - \eta a_2^2 &= a_2^2 + \frac{C_1(x)(c_2-d_2)}{4(1+3\alpha)} - \eta a_2^2 \\ &= (1-\eta)a_2^2 + \frac{C_1(x)(c_2-d_2)}{4(1+3\alpha)} \\ &= (1-\eta)\frac{(C_1(x))^3(c_2+d_2)}{2\left[(1+4\alpha)(C_1(x))^2 - (1+2\alpha)^2 C_2(x)\right]} + \frac{C_1(x)(c_2-d_2)}{4(1+3\alpha)} \\ &= (C_1(x))\left(\left[h(\eta) + \frac{1}{4(1+3\alpha)}\right]c_2 + \left[h(\eta) - \frac{1}{4(1+3\alpha)}\right]d_2\right), \end{aligned}$$

where

$$h(\eta) = \frac{(C_1(x))^2(1-\eta)}{2\left[(1+4\alpha)(C_1(x))^2 - (1+2\alpha)^2 C_2(x)\right]}.$$

Then, in view of (1.5), (1.6), and utilizing (2.8), we can conclude that

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{3x}{2(1+4\alpha)} & \text{if} \quad 0 \le |h(\eta)| \le \frac{1}{4(1+3\alpha)}, \\ 6x\,|h(\eta)| & \text{if} \quad |h(\eta)| \ge \frac{1}{4(1+3\alpha)}. \end{cases}$$

The proof of Theorem 3.1 is thus concluded. □

Following our previous discussion, our subsequent step involves introducing a corollary.

Corollary 3.1. *[25] Let f given by (1.1) be in the class $\mathcal{M}_\Sigma(0, \mathcal{B}(x,\xi))$. Then,*

$$|a_2| \le \frac{3x\sqrt{3x}}{\sqrt{\left|1-9x^2\right|}},$$

$$|a_3| \le \frac{27x^3}{\left|1-9x^2\right|} + \frac{3x}{2}$$

and

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{3x}{2} & \textit{if} \quad 0 \le |h_1(\eta)| \le \frac{1}{4}, \\ 6x\,|h_1(\eta)| & \textit{if} \quad |h_1(\eta)| \ge \frac{1}{4}, \end{cases}$$

where

$$h_1(\eta) = \frac{9x^2(1-\eta)}{2\left(1-9x^2\right)}.$$

4. Coefficient bounds of the class $\mathcal{H}_\Sigma(\alpha, \mu, \mathcal{B}(x, \xi))$

In this section, we introduce and define another distinct class of bi-univalent functions. Denoted as $\mathcal{H}_\Sigma(\alpha, \mu, \mathcal{B}(x, \xi))$, this new subclass enriches our comprehension and advances the domain of bi-univalent functions in mathematical analysis.

Definition 4.1. *A function* $f \in \Sigma$ *given by* (1.1), *with* $\alpha, \mu \in [0, 1]$ *and* $x \in (\frac{1}{2}, 1]$, *is said to be in the class* $\mathcal{H}_\Sigma(\alpha, \mu, \mathcal{B}(x, \xi))$ *if the following subordinations are satisfied*

$$(1 - \alpha + 2\mu)\frac{f(\xi)}{\xi} + (\alpha - 2\mu)f'(\xi) + \mu\xi f''(\xi) \prec \mathcal{B}(x, \xi) \tag{4.1}$$

and

$$(1 - \alpha + 2\mu)\frac{g(w)}{w} + (\alpha - 2\mu)g'(w) + \mu w g''(w) \prec \mathcal{B}(x, w), \tag{4.2}$$

where the function $g(w) = f^{-1}(w)$ *is defined by* (1.3) *and* $\mathcal{B}(x, \xi)$ *is the generating function of the Lucas-Balancing polynomials given by* (1.7).

Example 4.1. *A bi-univalent function* $f \in \Sigma$ *is said to be in the class* $\mathcal{H}_\Sigma(\alpha, 0, \mathcal{B}(x, \xi))$ *if the following subordination conditions hold:*

$$(1 - \alpha)\frac{f(\xi)}{\xi} + \alpha f'(\xi) \prec \mathcal{B}(x, \xi) \tag{4.3}$$

and

$$(1 - \alpha)\frac{g(w)}{w} + \alpha g'(w) \prec \mathcal{B}(x, w), \tag{4.4}$$

where the function $g = f^{-1}$ *is defined by* (1.3).

Example 4.2. *A bi-univalent function* $f \in \Sigma$ *is said to be in the class* $\mathcal{H}_\Sigma(1, 0, \mathcal{B}(x, \xi))$ *if the following subordination conditions hold:*

$$f'(\xi) \prec \mathcal{B}(x, \xi) \tag{4.5}$$

and

$$g'(w) \prec \mathcal{B}(x, w), \tag{4.6}$$

where the function $g = f^{-1}$ *is defined by* (1.3).

Theorem 4.1. *Let* $f \in \Sigma$ *of the form (1.1) be in the class* $\mathcal{H}_\Sigma(\alpha, \mu, \mathcal{B}(x, \xi))$. *Then,*

$$|a_2| \leq \frac{3x\sqrt{3x}}{\sqrt{\left|9x^2(1 + 2\alpha + 2\mu) - (18x^2 - 1)(1 + \alpha)^2\right|}}$$

and

$$|a_3| \leq \frac{27x^3}{\left|9x^2(1 + 2\alpha + 2\mu) - (18x^2 - 1)(1 + \alpha)^2\right|} + \frac{3x}{(1 + 2\alpha + 2\mu)}.$$

Proof. Assuming f belongs to $\mathcal{H}_{\Sigma}(\alpha, \mu, \mathcal{B}(x, \xi))$, where $0 \leq \alpha, \mu \leq 1$, Eqs (4.1) and (4.2) imply that

$$(1-\alpha+2\mu)\frac{f(\xi)}{\xi}+(\alpha-2\mu)f'(\xi)+\mu\xi f''(\xi)=\mathcal{B}(x,u(\xi)) \tag{4.7}$$

and

$$(1-\alpha+2\mu)\frac{g(w)}{w}+(\alpha-2\mu)g'(w)+\mu w g''(w)=\mathcal{B}(x,v(w)), \tag{4.8}$$

where $g(w) = f^{-1}(w)$ and $u,\ v \in \Omega$ are defined in (2.7).

Upon substituting the definition of $\mathcal{B}(x, \xi)$ from (1.7) into the right-hand sides of Eqs (4.7) and (4.8), we obtain

$$\begin{aligned}\mathcal{B}(x,u(\xi)) &= 1+C_1(x)c_1\xi+\left[C_1(x)c_2+C_2(x)c_1^2\right]\xi^2\\ &\quad+\left[C_1(x)c_3+2C_2(x)c_1c_2+C_3(x)c_1^3\right]\xi^3+\cdots\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\mathcal{B}(x,v(w)) &= 1+C_1(x)d_1w+\left[C_1(x)d_2+C_2(x)d_1^2\right]w^2\\ &\quad+\left[C_1(x)d_3+2C_2(x)d_1d_2+C_3(x)d_1^3\right]w^3+\cdots.\end{aligned} \tag{4.10}$$

Hence, Eqs (4.7) and (4.8) become

$$\begin{aligned}&(1-\alpha+2\mu)(1+a_2\xi+a_3\xi^2+a_4\xi^3+\cdots)\\ &\quad+(\alpha-2\mu)(1+2a_2\xi+3a_3\xi^2+4a_4\xi^3+\cdots)\\ &\quad+\mu\xi(2a_2+6a_3\xi+12a_4\xi^2+\cdots)\\ &\quad=1+C_1(x)c_1\xi+\left[C_1(x)c_2+C_2(x)c_1^2\right]\xi^2+\left[C_1(x)c_3+2C_2(x)c_1c_2+C_3(x)c_1^3\right]\xi^3+\cdots\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}&(1-\alpha+2\mu)(1-a_2w+(2a_2^2-a_3)w^2-(5a_2^3-5a_2a_3+a_4)w^3+\cdots)\\ &\quad+(\alpha-2\mu)(1-2a_2w+3(2a_2^2-a_3)w^2-4(5a_2^3-5a_2a_3+a_4)w^3+\cdots)\\ &\quad+\mu\xi(-2a_2+6(2a_2^2-a_3)w-12(5a_2^3-5a_2a_3+a_4)w^2+\cdots)\\ &\quad=1+C_1(x)d_1w+\left[C_1(x)d_2+C_2(x)d_1^2\right]w^2+\left[C_1(x)d_3+2C_2(x)d_1d_2+C_3(x)d_1^3\right]w^3+\cdots.\end{aligned} \tag{4.12}$$

When equating the coefficients in Eqs (4.11) and (4.12), we get

$$(1+\alpha)a_2=C_1(x)c_1, \tag{4.13}$$

$$(1+2\alpha+2\mu)a_3=C_1(x)c_2+C_2(x)c_1^2, \tag{4.14}$$

$$-(1+\alpha)a_2=C_1(x)d_1 \tag{4.15}$$

and

$$2(1+2\alpha+2\mu)a_2^2-(1+2\alpha+2\mu)a_3=C_1(x)d_2+C_2(x)d_1^2. \tag{4.16}$$

With the utilization of (4.13) and (4.15), we derive the following equations

$$c_1 = -d_1 \tag{4.17}$$

and

$$c_1^2 + d_1^2 = \frac{2(1+\alpha)^2 a_2^2}{(C_1(x))^2}. \tag{4.18}$$

Additionally, applying Eqs (4.14), (4.16) and (4.18) results in

$$a_2^2 = \frac{(C_1(x))^3 (c_2 + d_2)}{2\left[(1+2\alpha+2\mu)(C_1(x))^2 - (1+\alpha)^2 C_2(x)\right]}. \tag{4.19}$$

By employing Lemma 1.2 and analyzing Eqs (4.13) and (4.17), we can deduce

$$|a_2|^2 \leq \frac{|C_1(x)|^3}{\left|(1+2\alpha+2\mu)(C_1(x))^2 - (1+\alpha)^2 C_2(x)\right|}, \tag{4.20}$$

therefore

$$|a_2| \leq \frac{|C_1(x)| \sqrt{|C_1(x)|}}{\sqrt{\left|(1+2\alpha+2\mu)(C_1(x))^2 - (1+\alpha)^2 C_2(x)\right|}}. \tag{4.21}$$

When substituting $C_1(x)$ and $C_2(x)$ as provided in (1.5) and (1.6) into Eq (4.21), it results in the following expression

$$|a_2| \leq \frac{3x\sqrt{3x}}{\sqrt{\left|9x^2(1+2\alpha+2\mu) - (18x^2-1)(1+\alpha)^2\right|}}.$$

By subtracting Eq (4.16) from Eq (4.14), we obtain:

$$a_3 = a_2^2 + \frac{C_1(x)(c_2 - d_2)}{2(1+2\alpha+2\mu)}. \tag{4.22}$$

Consequently, this results in the following inequality

$$|a_3| \leq |a_2|^2 + \frac{|C_1(x)|\,|c_2 - d_2|}{2(1+2\alpha+2\mu)}. \tag{4.23}$$

By employing Lemma 1.2 and utilizing (1.5) and (1.6), we obtain

$$|a_3| \leq \frac{27x^3}{\left|9x^2(1+2\alpha+2\mu) - (18x^2-1)(1+\alpha)^2\right|} + \frac{3x}{(1+2\alpha+2\mu)}. \tag{4.24}$$

The proof of Theorem 4.1 is thus concluded. □

5. Fekete-Szegö functional estimations of the class $\mathcal{H}_\Sigma(\alpha,\mu,\mathcal{B}(x,\xi))$

In this section, the utilization of the values of a_2^2 and a_3 assists in deriving the Fekete-Szegö inequality applicable to functions $f\in\mathcal{H}_\Sigma(\alpha,\mu,\mathcal{B}(x,\xi))$.

Theorem 5.1. *Let $f\in\Sigma$ given by the form (1.1) be in the class $\mathcal{H}_\Sigma(\alpha,\mu,\mathcal{B}(x,\xi))$. Then,*

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\frac{3x}{1+2\alpha+2\mu} & \text{if}\quad 0\le|h(\eta)|\le\frac{1}{2(1+2\alpha+2\mu)},\\ 6x\,|h(\eta)| & \text{if}\quad |h(\eta)|\ge\frac{1}{2(1+2\alpha+2\mu)},\end{cases}$$

where

$$h(\eta)=\frac{9x^2(1-\eta)}{2\left[9x^2(1+2\alpha+2\mu)-(18x^2-1)(1+\alpha)^2\right]}.$$

Proof. Equations (4.19) and (4.22) yield

$$\begin{aligned}a_3-\eta a_2^2&=a_2^2+\frac{C_1(x)(c_2-d_2)}{2(1+2\alpha+2\mu)}-\eta a_2^2\\&=(1-\eta)a_2^2+\frac{C_1(x)(c_2-d_2)}{2(1+2\alpha+2\mu)}\\&=(1-\eta)\frac{(C_1(x))^3(c_2+d_2)}{2\left[(1+2\alpha+2\mu)(C_1(x))^2-(1+\alpha)^2C_2(x)\right]}+\frac{C_1(x)(c_2-d_2)}{2(1+2\alpha+2\mu)}\\&=(C_1(x))\left(\left[h(\eta)+\frac{1}{2(1+2\alpha+2\mu)}\right]c_2+\left[h(\eta)-\frac{1}{2(1+2\alpha+2\mu)}\right]d_2\right),\end{aligned}$$

where

$$h(\eta)=\frac{(C_1(x))^2(1-\eta)}{2\left[(1+2\alpha+2\mu)(C_1(x))^2-(1+\alpha)^2C_2(x)\right]}.$$

Considering (1.5), (1.6) and applying (2.8), we can deduce that

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\frac{3x}{1+2\alpha+2\mu} & \text{if}\quad 0\le|h(\eta)|\le\frac{1}{2(1+2\alpha+2\mu)},\\ 6x\,|h(\eta)| & \text{if}\quad |h(\eta)|\ge\frac{1}{2(1+2\alpha+2\mu)}.\end{cases}$$

The proof of Theorem 5.1 is thus concluded. □

Corollary 5.1. *Let $f\in\Sigma$ given by the form (1.1) be in the class $\mathcal{H}_\Sigma(\alpha,0,\mathcal{B}(x,\xi))$. Then,*

$$|a_2|\le\frac{3x\sqrt{3x}}{\sqrt{\left|9x^2(1+2\alpha)-(18x^2-1)(1+\alpha)^2\right|}},$$

$$|a_3|\le\frac{27x^3}{\left|9x^2(1+2\alpha)-(18x^2-1)(1+\alpha)^2\right|}+\frac{3x}{1+2\alpha}$$

and

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\frac{3x}{1+2\alpha} & \text{if}\quad 0\le|h_2(\eta)|\le\frac{1}{2(1+2\alpha)},\\ 6x\,|h_2(\eta)| & \text{if}\quad |h_2(\eta)|\ge\frac{1}{2(1+2\alpha)},\end{cases}$$

where

$$h_2(\eta)=\frac{9x^2(1-\eta)}{2\left[9x^2(1+2\alpha)-(18x^2-1)(1+\alpha)^2\right]}.$$

Corollary 5.2. *Let $f \in \Sigma$ given by the form (1.1) be in the class $\mathcal{H}_\Sigma(1, 0, \mathcal{B}(x, \xi))$. Then*

$$|a_2| \le \frac{3x\sqrt{3x}}{\sqrt{\left|4 - 45x^2\right|}},$$

$$|a_3| \le \frac{27x^3}{\left|4 - 45x^2\right|} + \frac{x}{3}$$

and

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{x}{3} & if \quad 0 \le |h_3(\eta)| \le \frac{1}{6}, \\ 6x\,|h_3(\eta)| & if \quad |h_3(\eta)| \ge \frac{1}{6}, \end{cases}$$

where

$$h_3(\eta) = \frac{9x^2(1-\eta)}{2(4 - 45x^2)}.$$

6. Conclusions

We introduced two novel subclasses of bi-univalent functions within the open unit disk $\mathbb{U}$, namely $\mathcal{M}_\Sigma(\alpha, \mathcal{B}(x, \xi))$ and $\mathcal{H}_\Sigma(\alpha, \mu, \mathcal{B}(x, \xi))$, employing Lucas-Balancing polynomials. Our investigation delves into the initial estimates of the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Furthermore, by utilizing of a_2^2 and a_3 a crucial tool, we established the Fekete-Szegö inequalities $\left|a_3 - \eta a_2^2\right|$ for functions belonging to $\mathcal{M}_\Sigma(\alpha, \mathcal{B}(x, \xi))$ and $\mathcal{H}_\Sigma(\alpha, \mu, \mathcal{B}(x, \xi))$.

Moreover, by appropriately specializing the parameter, we obtained new results for the subclasses $\mathcal{M}_\Sigma(0, \mathcal{B}(x, \xi))$, $\mathcal{H}_\Sigma(\alpha, 0, \mathcal{B}(x, \xi))$, and $\mathcal{H}_\Sigma(1, 0, \mathcal{B}(x, \xi))$, defined in Examples (2.1), (4.1), and (4.2), respectively. These results establish connections between these subclasses and the Lucas-Balancing Polynomials. Utilizing these subclasses, we derive estimations for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, and investigate the Fekete-Szegö inequalities.

Author contributions

A. H., M. M. and A. A.: Conceptualization; A. H. and M. M.: Data curation; A. H. and A.A.: Formal analysis; A. H., M. M. and A. A.: Investigation; A. H. and M. M.: Methodology; A. H. and M. M.: Resources; A. H., M. M. and A. A.: Validation; A. H., M. M. and A. A.: Visualization; A. H. and A. A.: Writing original draft; A. H. and A. A.: Writing review & editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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